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ON THE MAX PARETO POWER SERIES DISTRIBUTION

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Abstract: *This paper presents a new distribution wich unitary treated through the family of the power series distributions, resulting in a new distribution which we called a Max Pareto power series (MaxParPS) distribution. Some properties and reliability characteristics (e.g. survival function, hazard rate) are studied. It also shows that in some conditions a Poisson limit theorem for this type of distribution takes place.*

Keywords: *distribution of the maximum, limit theorem, Pareto distribution, power series distribution.*
MSC2010: 60K10, 62N05.

1. INTRODUCTION

Together the most common reliability distributions (exponential, Erlang, Weibull), there was also Pareto distribution which has applications not only in the economy (e.g. income of the population), but also to the study of lifetime of the systems k of n, as well as in field of quality assurance [5].

Pareto at the end XIXth century, formulated the principle of 80/20, of the unbalanced distributions, which postulates that 80% of the effects are generated by 20% of the cases.

This is the reason for introducing this new class of MaxParPS distribution with the aim of study the probability behavior of the most complicated processes.

The methodology and techniques of working are presented and analyzed in [2], which allows the study of the distribution of the maximum of a random sample of size Z of the statistical population that has Pareto distribution. Random variable (r.v.) Z has a distribution that is part of the power series distributions class (PSD, [1]).

The general problem of determining the distribution of the maximum and minimum of a random sequence will was solved by Louzada et al. in [3], with working tool of the generate function composing of the number of the r.v. of sequence with the survival function of the r.v. components of the sequence.

Instead, in this paper is approached, in a unitary manner, the distribution of the maximum of a sequence of independent Pareto distributed r.v. through the PSD class, distribution enjoyed by the number of the r.v. of the sequence.

The case of the minim was analyzed in [4] when it was discussed about the lifetime Min Pareto power series (MinParPS) distribution.

2. THE MAXPARPS DISTRIBUTION

We consider the r.v. $X_i \sim Par(\mu,\alpha)$, $\mu,\alpha>0$, where $(X_i)_{i\geq 1}$ are independent and identically distributed random variables (i.i.d.r.v.), with the distribution function (d.f.) $F_{X_i}(x)=F_{Par}(x)=1-\left(\frac{\mu}{x}\right)^{\alpha}$, $x\geq\mu$ and the probability density function (p.d.f.) $f_{X_i}(x)=f_{Par}(x)=\frac{\alpha\mu^{\alpha}}{x^{\alpha+1}}$, $x\geq\mu$.

Also, we note by $U_{Par}=\max\{X_1,X_2,...,X_N\}$, where the r.v. $N\in PSD$, [1,2].

Proposition 2.1. *If r.v. $U_{Par}=\max\{X_1,X_2,...,X_N\}$, where $(X_i)_{i\geq 1}$ are i.i.d.r.v., $X_i\sim Par(\mu,\alpha)$, $\mu,\alpha>0$ and $N\in PSD$, with $P(N=n)=\frac{a_n\theta^n}{A(\theta)}$, $n=1,2,...,$ $\theta\in(0,\tau),\tau>0$, r.v. $(X_i)_{i\geq 1}$ and N being independent, then the d.f. of the r.v. U_{Par} is given by:*

$$U_{Par}(x)=\frac{A\left[\theta-\theta\left(\frac{\mu}{x}\right)^{\alpha}\right]}{A(\theta)}, x\geq\mu. \qquad (2.1)$$

We note that r.v. U_{Par} following the MaxParPS distribution of parameters μ,α and θ by $U_{Par}\sim MaxParPS(\mu,\alpha,\theta)$.

Consquence 2.1. *The survival function of the r.v. U_{Par} is the following:*

$$S_{U_{Par}}(x)=\frac{A(\theta)-A\left[\theta-\theta\left(\frac{\mu}{x}\right)^{\alpha}\right]}{A(\theta)}, \quad x\geq\mu.$$

Consequence 2.2. *P.d.f. of the r.v. U_{Par} is characterized by the relationship:*

$$u_{Par}(x)=\frac{\alpha\theta\mu^{\alpha}A'\left[\theta-\theta\left(\frac{\mu}{x}\right)^{\alpha}\right]}{x^{\alpha+1}A(\theta)}, \qquad (2.2)$$

for any $x\geq\mu$.

Proposition 2.2. *Hazard rate of the r.v. U_{Par} is given by:*

$$h_{U_{Par}}(x)=\frac{u_{Par}(x)}{1-U_{Par}(x)}$$

$$=\frac{\alpha\theta\mu^{\alpha}A'\left[\theta-\theta\left(\frac{\mu}{x}\right)^{\alpha}\right]}{x^{\alpha+1}\left[A(\theta)-A\left[\theta-\theta\left(\frac{\mu}{x}\right)^{\alpha}\right]\right]}, \quad x\geq\mu.$$

Proposition 2.3. *If $(X_i)_{i\geq 1}$ is a sequence of independent Pareto distributed r.v., nonnegative, absolutely continous type, with the d.f. F_{Par} and $N\in PSD$, $P(N=n)=\frac{a_n\theta^n}{A(\theta)}$, $n=1,2,...,$ $(a_n)_{n\geq 1}$ a sequence of nonnegative real number, $A(\theta)=\sum_{n\geq 1}a_n\theta^n$, $\forall\ \theta\in(0,\tau),\tau>0$, then:*

$$\lim_{\theta\to 0^+}U_{Par}(x)=\left(F_{Par}(x)\right)^k, \quad x\geq\mu,$$

where $k=\min\{n\in N^*, a_n>0\}$.

Proof: By applying the l'Hospital rule k-time, we have:

$$\lim_{\theta\to 0^+} U_{Par}(x) = \lim_{\theta\to 0^+} \frac{A\left[\theta-\theta\left(\frac{\mu}{x}\right)^{\alpha}\right]}{A(\theta)}$$

$$\overset{\frac{0}{0}}{=} \lim_{\theta\to 0^+} \frac{\left(1-\left(\frac{\mu}{x}\right)^{\alpha}\right)^{k} A^{(k)}\left[\theta-\theta\left(\frac{\mu}{x}\right)^{\alpha}\right]}{A^{(k)}(\theta)}$$

$$= \frac{\left(1-\left(\frac{\mu}{x}\right)^{\alpha}\right)^{k} k!a_k}{k!a_k} = \left(1-\left(\frac{\mu}{x}\right)^{\alpha}\right)^{k}, \quad x \geq \mu.$$

□

Consequence 2.3. *The r^{th} moments, $r \in N$, $r \geq 1$ of the r.v. $U_{Par} \sim MaxParPS(\mu,\alpha,\theta)$ is the following:*

$$EU_{Par}^{r} = = \sum_{n\geq 1} \frac{a_n\theta^n}{A(\theta)} \cdot E\left[\max\{X_1,X_2,...,X_N\}\right]^r, \quad (2.3)$$

where p.d.f. of the r.v. $\max\{X_1,X_2,...,X_N\}$ *is given by* $f_{\max\{X_1,X_2,...,X_N\}}(x) = zf_{Par}(x)(F_{Par}(x))^{n-1}$.

Proof: It is known that the distribution function of the maximum of a sample of size $N = n$ which has the d.f. F_{Par} is $U_n(x) = (F_{Par}(x))^n$.

With total probability formula, a d.f. of the maximum of a sequence of i.i.d.r.v. in a random number, has the expression:

$$U_{Par}(x) = \sum_{n\geq 1} U_n(x)\cdot P(N=n) = \sum_{n\geq 1} (F_{Par}(x))^n \cdot P(N=n).$$

By determining the derivative of the latter relation in connection with x, we obtain:

$$u_{Par}(x) = = \sum_{n\geq 1} n f_{Par}(x)(F_{Par}(x))^{n-1} \cdot P(N=n) \quad (2.4)$$

where $z f_{Par}(x)(F_{Par}(x))^{n-1}$ is the p.d.f. of the r.v. $\max\{X_1,X_2,...,X_N\}$. By applying the mean of the relationship (2.4), the relationship (2.3) is obtained.

□

3. POISSON LIMIT THEOREM FOR MAXPARPS DISTRIBUTION

The study of special cases of distributions MaxParPS is necessary for this section because we want to study under what conditions the Max-Pareto-Binomial zero truncated (MaxParB) distribution, respectively, Max-Pareto-Poisson zero truncated (MaxParP) distribution are approximate.

The d.f. of the MaxParB is defined by (2.1), where $N \sim Binom^*(n,p) \in PSD$, $n \in \{1,2,...\}$, $p \in (0,1)$ and $A(\theta) = (1+\theta)^n - 1$, where $\theta = \frac{p}{1-p}$, namely:

$$U_{ParB}(x) = \frac{\left(1-p\left(\frac{\mu}{x}\right)^{\alpha}\right)^{n} - (1-p)^n}{1-(1-p)^n}, \quad (2.5)$$

for any $x \geq \mu$.

A r.v. which admits the d.f. according to the relationship (2.5), is denoted by

$U_{ParB} \sim MaxParB(\mu,\alpha,n,p)$, $\mu,\alpha>0$, $n\in\{1,2,...\}$, and $p\in(0,1)$.

P.d.f. of the r.v. U_{ParB} is defined according to the relationship (2.2), so that :

$$u_{ParB}(x)=\frac{np\alpha\mu^{\alpha}\left(1-p\left(\frac{\mu}{x}\right)^{\alpha}\right)^{n-1}}{x^{\alpha+1}\left(1-(1-p)^{n}\right)}, \quad (2.6)$$

for any $x\geq\mu$.

At the same time, the MaxParP distribution is characterized by the d.f. defined by the relationship (2.1), where $N\sim Poisson*(\lambda)\in PSD,\ \lambda>0$ and $A(\theta^*)=e^{\theta^*}-1$, with $\theta^*=\lambda$:

$$U_{ParP}(x)=\frac{e^{-\lambda\left(\frac{\mu}{x}\right)^{\alpha}}-e^{-\lambda}}{1-e^{-\lambda}},\ x\geq\mu. \quad (2.7)$$

and p.d.f.. according to the relationship (2.2):

$$u_{ParP}(x)=\frac{\alpha\lambda\mu^{\alpha}e^{-\lambda\left(\frac{\mu}{x}\right)^{\alpha}}}{x^{\alpha+1}\left(1-e^{-\lambda}\right)},\ x\geq\mu. \quad (2.8)$$

D.f. of (2.7) characterizes the distribution of the r.v. will we note thus: $U_{ParP}\sim MaxParP(\mu,\alpha,\lambda)$, $\mu,\alpha,\lambda>0$.

Under the above conditions, takes place the following result:

Theorem 3.1. (Poisson Limit Theorem) *If r.v. $U_{ParB}\sim MaxParB(\mu,\alpha,n,p)$, with $n\to\infty$ and $p\to 0^+$ such that $n\cdot p\to\lambda$, $\lambda>0$, then:*

$$\lim_{\substack{n\to\infty\\ p\to 0^+}} U_{ParB}(x)=U_{ParP}(x),\ \forall x\geq\mu,$$

where $U_{ParB}(x)$, respectively $U_{ParP}(x)$, $x\geq\mu$ are d.f. of the r.v. $U_{ParB}\sim MaxParB(\mu,\alpha,n,p)$, respectively $U_{ParP}\sim MaxParP(\mu,\alpha,\lambda)$, defined by the relationship (2.5) and (2.7).

Proof. By calculating separately two elementary limits:

$$\lim_{\substack{n\to\infty\\ p\to 0^+}}(1-p)^n=\lim_{\substack{n\to\infty\\ p\to 0^+}}\left[(1-p)^{-1/p}\right]^{-pn}=e^{-\lambda}$$

and

$$\lim_{\substack{n\to\infty\\ p\to 0^+}}\left(1-p\left(\frac{\mu}{x}\right)^{\alpha}\right)^{n}=e^{-\lim\limits_{\substack{n\to\infty\\ p\to 0^+}} p\left(\frac{\mu}{x}\right)^{\alpha}n}=e^{-\lambda\left(\frac{\mu}{x}\right)^{\alpha}},$$

we have the following final limit:

$$\lim_{\substack{n\to\infty\\ p\to 0^+}}U_{ParB}(x)=$$

$$=\lim_{\substack{n\to\infty\\ p\to 0^+}}\frac{\left(1-p\left(\frac{\mu}{x}\right)^{\alpha}\right)^{n}-(1-p)^n}{1-(1-p)^n}$$

$$=\frac{e^{-\lambda\left(\frac{\mu}{x}\right)^{\alpha}}-e^{-\lambda}}{1-e^{-\lambda}}=U_{ParP}(x),\ \forall x\geq\mu.$$

□

Remark 3.1. Poisson limit theorem for the distribution MaxParPS is confirmed visually of the plot in Figure 1, where they are presented p.d.f. of the MaxParB and MaxParP distributions for the following parameters: $\mu = \alpha = 1$, $n = 20$, $\lambda = 10$ and $p = 1/2$.

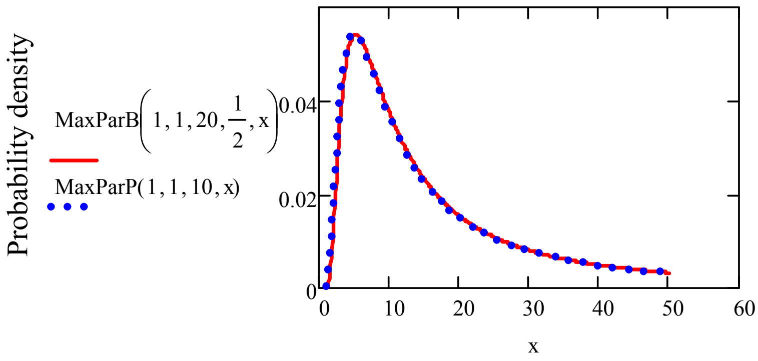

Fig.1. P.d.f. of the MaxParB and MaxParP distributions

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