STOCHASTIC APPROXIMATION FOR RELIABILITY PROBLEMS

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Abstract: *The article discusses the mathematical model for the stochastic approximation of reliability problems and an optimality system for the stochastic approximation plan given by M.T. Wasan. The end of the paper defines the concept of adapted stochastic approximation plan (SA).*

Keywords: *stochastic approximation, random variables, reliability, probability*

1. INTRODUCTION

In many probability theory applications the random variables occur and they depend on one or more parameters. Thus, in the reliability theory, there is the problem of the time variation study of the characteristics of a certain system (automobile, computer or any type of machine, etc.), time being in this case a parameter.

Such situations can be studied with the help of stochastic processes, these representing a current research direction of the probability theory, just because of the fact that they have new and numerous practical uses.

Stochastic approximation algorithms, introduced by H. Robbins and S. Monro since 1950 have been the subject of rich literature due to its wide applicability area.

A field where stochastic approximation algorithms is present is the reliability theory, whose purpose is to determine the laws of emergence of the problems in a system, with the probability of fulfilling the functions with certain performances and without problems, in a certain time interval and specified exploitation conditions.

2. THE ENUNCIATION OF THE GENERAL PROBLEM OF STOCHASTIC APPROXIMATION

A stochastic approximation problem may be synthetized as following:

Take an experiment whose results depend on a t *variable and mark* $Y(t)$ *the random variable corresponding to those results.*

Pick a random value t_1 *that leads to the observed value* $Y(t_1)$ *of the random variable* $Y(t)$.

Mark $F(t) = EY(t)$, E *being the random variable average* $Y(t)$.

The result should be the determination of a $t*$ *value to verify the equation* $F(t*) = \alpha$, *where* $\alpha \in R$ *randomly chosen.*

The solution of this problem will be found with an iterative procedure.

Choose a non-rising series of positive numbers $(a_n)_n$ $(a_n \geq a_{n+1}, \forall n \in N)$ and establish a recursion, for the purpose of selecting another t value, for the following experiment:

$$t_{n+1} = t_n - a_n[y(t_n) - \alpha] \tag{2.1}$$

An *n* number of experiments is assumed to have been made, obtaining a result, therefore the values of t_n and $y(t_n)$ are already known.

Then, using the recursion relation (2.1) the value of t used in the number (n+1) experiment can be determined.

If in the relation (2.1) $\alpha = 0$ is chosen, this is reduced to

$$t_{n+1} = t_n - a_n y(t_n) \tag{2.2}$$

In the relation (2.2) if $y(t_n) > 0$ then $t_{n+1} < t_n$, and if $y(t_n) < 0$ then $t_{n+1} > t_n$.

These observations are important for the solution in the equation $F(t^*) = \alpha$.

The conclusion is that if $y(t_n) > 0$, then the t value for the (n+1) step should drop; if $y(t_n) < 0$, then the chosen t for the (n+1) step should rise.

What is interesting is the condition that the array $(t_n)_n$ should converge in a mean square value, with probability 1, for the t^* value.

3. REAL STOCHASTIC APPROXIMATION FOR RELIABILITY PROBLEMS

We analyze the functionality of a "system" that has a certain t "lifespan", with a corresponding $F(t)$ distribution function.

The system is examined at $t_1, t_2, ..., t_n$ time. If the inspection reveals the fact that the robot is not operative, then it is repaired (if possible) or replaced (if it cannot be fixed).

The general problem is the selection of an inspection plan $t_1, t_2, ..., t_n$, in an optimum way.

Next, an optimality criterion is defined and it is proven that the stochastic approximation plan fulfills this criterion.

If the system's "lifespan" is a continuous random variable, with an exponential repair, then the distribution function becomes:

$$F(t) = \begin{cases} 0, & if < 0 \\ 1 - e^{-\lambda t}, if\ t \geq 0 \end{cases} \tag{3.1}$$

where $\lambda > 0$ is an unknown parameter with a known positive interval, that is $\lambda \in (\lambda_1, \lambda_2)$ with $0 < \lambda_1 < \lambda_2$ given.

The distribution function is differentiable for any $t > 0$ and $F'(t) = \rho(t)$ is the probability density.

Mark $T_i = t_i - t_{i-1}$, $i = 2,3,....$, $T_1 = t_1$, the time interval between two consecutive inspections.

Take the array $\{t_n\}_n$ as an arbitrary array of random variables, having the distribution depending on a real parameter, different from λ.

Consider a random variables array $(Y_n)_n$, defined as:

$$Y_i = \begin{cases} 1, & with\ probability\ e^{-\lambda T_i} \\ 0, & with\ probability\ 1 - e^{-\lambda T_i} \end{cases} \tag{3.2}$$

That is, if at moment t_i verification, the system is not functional, then $Y_i = 0$, or $Y_i = 1$.

Take f_n a measurable function, of *n* variables $Y_1, Y_2, ..., Y_n$, with real values which does not depend on λ.

The time intervals between two consecutive inspections are given by the following relations:

$$T_1 = \max\{0, v_1\}$$

$$T_{n+1} = \max\{0, f_n(Y_1, Y_2, ..., Y_n) + v_n\},\ n = 1,2,... \tag{3.3}$$

Intuitively, after the *n* number of inspections, the next inspection, defined by T_{n+1}, depends on the past observations $(Y_1, Y_2, ..., Y_n)$ with f_n.

M.T. Wasan suggests an optimality criterion that takes into consideration the information about λ, this becoming unavailable after inspection procedures.
The information average is defined after applying a I plan of *n* inspections with the relation:

$$J_n(I,\lambda) = n^{-1}E\left[\frac{d}{d\lambda}\log L_n(\lambda)\right]^2, \tag{3.4}$$

where $L_n(\lambda)$ is the probability function of λ when $(Y_1, Y_2, ..., Y_n)$, respectively $(T_1, T_2, ..., T_n)$ are known

Then,

$$J(I,\lambda) = \liminf_{n\to\infty} J_n(I,\lambda), \tag{3.5}$$

is called the medium limit of the information obtained after a I plan.
The next problem is that of maximizing $J_n(I,\lambda)$ and $J(I,\lambda)$ after a correct selection of the I plan.
The following theorem gives a result, concerning the efficient estimation of λ.

<u>Theorem 3.1</u>
For each value of n,

$$J_n(I,\lambda) \le \lambda^{-1}\cdot T_\lambda(2-\lambda T_\lambda), \ \forall \lambda, I \tag{3.6}$$

where T_λ represents the solution of the equation

$$e^{-\lambda T} = 1 - \frac{1}{2}\lambda T \tag{3.7}$$

Observation: $T_\lambda = -\dfrac{\log p}{\lambda}$ and T_λ is the value of the distribution function in $x = 100(1-p)$, of the exponentially distributed variable, where $p \approx 0{,}203$.

<u>Demonstration</u>
The probability conditioned by the observations $Y_1, Y_2, ..., Y_n$, when $(v_1, v_2, ..., v_n)$ are known is:

$$\prod_{i=1}^{n}\left(e^{-\lambda T_i}\right)^{Y_i}\left(1-e^{-\lambda T_i}\right)^{1-Y_i} \tag{3.8}$$

As the distribution of $(v_1, v_2, ..., v_n)$ is independent of λ, then,

$$\frac{d}{d\lambda}\log L_n(\lambda) = -\sum_{i=1}^{n} T_i Y_i + \sum_{i=1}^{n}\frac{(1-Y_i)T_i e^{-\lambda T_i}}{(1-e^{-\lambda T_i})} =$$

$$= -\sum_{i=1}^{n}\frac{T_i(Y_i - e^{-\lambda T_i})}{(1-e^{-\lambda T_i})} \tag{3.9}$$

Mark $\dfrac{T_i(Y_i - e^{-\lambda T_i})}{1-e^{-\lambda T_i}} = X_i$

Using the relation (3.2) the result is

$$E(X_i X_j) = E\left\{X_i T_j\left(1-e^{-\lambda T_j}\right)^{-1}\cdot E\left[\left(Y_j - e^{-\lambda T_j}\right)\left(Y_1, ..., Y_{j-1}, T_1, ..., T_j\right)\right]\right\} = 0, \text{ for } j > i \tag{3.11}$$

Further on, the result is

$$E\left[\frac{d}{d\lambda}\log L_n(\lambda)\right]^2 = \sum_{i=1}^{n}E(X_i^2) = \sum_{i=1}^{n}E\left\{T_i^2\left(1-e^{-\lambda T_i}\right)^{-2}\cdot E\left[\left(Y_i - e^{-\lambda T_i}\right)^2 \middle| Y_1, ..., Y_{i-1}, T_1, ..., T_i\right]\right\} =$$

$$= E\sum_{i=1}^{n}T_i^2\left(1-e^{-\lambda T_i}\right)^{-1}e^{-\lambda T_i} \le nT_\lambda^2\left(1-e^{-\lambda T_\lambda}\right)^{-1}e^{-\lambda T_\lambda} \tag{3.11}$$

which shows that the function $T^2\left(1-e^{-\lambda T}\right)^{-1}e^{-\lambda T}$ is maximized for $T = T_\lambda$, hence it results the required enunciation.
Equality is obtained in (3.11) only and only if $p(T_i = T_\lambda) = 1, \forall i$.

If λ is known, then the inspection plan for the purpose of maximizing $J_n(I,\lambda)$, for each n and λ is called a periodical inspection plan with a T_λ time interval between two consecutive inspections. Aside from an array of the unknown values of λ, there is no optimal plan to obtain equality for.

Defining the adapted stochastic approximation plan

Definition 3.1

_ A I inspection plan is called adapted (related to $J(I,\lambda)$) if the following equality occurs:

$$J(I,\lambda) = \lambda^{-1} \cdot T_\lambda (2 - \lambda T_\lambda) \tag{3.12}$$

The following plan called the stochastic approximation plan (SA) is based on the Robins-Monro method and uses the fact that T_λ corresponds to the $100(1-p) \approx 79{,}7$ value of the exponential distribution, independent of λ.

If the T_1 is selected in the $\left[T_{\lambda_1}, T_\lambda\right]$ interval and the $(T_1, T_2, ..., T_n)$ is considered to be defined, then the following can be defined:

$$\lambda_n = T_n^{-1} \log p$$
$$A_n = \lambda_n^{-1} \cdot p^{-1} = -T_n (p \log p)^{-1}$$
$$T_{n+1} = \max\left[T_{\lambda_2}, \min\left\{T_\lambda, T_n + n^{-1} \cdot A_n (y_n - p)\right\}\right], \text{ for } n = 1,2 \tag{3.13}$$

Wasan establishes the following result with the condition that the defined relations are fulfilled.

Theorem 3.2

The SA plan described by the relations (3.13) is adapted (the demonstration is based on the convergence verification in the probability of the array $(T_n)_n$ to T_λ).

This result represents one of the usages of the stochastic approximation method for the resolution of applied problems.

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