DERIVATION OF MASS, STIFFNESS AND DAMPING MATRICES OF A MODEL WING FROM EXPERIMENTAL MODAL DATA

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Abstract: *This paper presents the results of the experimental modal analysis of a wing structure along with the derivation of full mass, damping and stiffness matrices from modal parameters. Using the derived matrices, the natural frequencies of the structure were computed and compared with the experimentally determined ones. Very good correlation was found, which confirms the validity of the method.*

Keywords: *experimental modal analysis, mass, stiffness, damping, mode shapes, frequencies*

1. INTRODUCTION

The specialized systems for experimental modal analysis do not usually provide mass, stiffness and damping matrices in spatial coordinates but the ones in the modal space. While the modal matrices are very useful in a large number of engineering applications, there are cases in which matrices in spatial coordinates are preferred. Such a case is the active control of aeroelastic oscillations and flutter phenomena in which it would be difficult to express the aerodynamic forces in the modal space. To avoid such complications and to use the measured modal data, a method of deriving these matrices in spatial coordinates from the ones in modal space is needed. Such a method is well presented in [2] and is applied in our study on experimental modal data obtained for a model wing aimed for flutter control experiments in aerodynamic wind tunnel.

More than the derivation of mass, stiffness and damping matrices in spatial coordinates from modal data, this paper aims to validate the obtained matrices. To accomplish that, the natural frequencies of the free undamped wing structure are computed by using the derived mass and stiffness matrices. Very good correlation is found between the computed and the experimentally determined natural frequencies.

2. EQUIPMENT USED AND TEST CONFIGURATION

The Prodera installation is a modal analysis equipment that uses up to 16 unidirectional accelerometers simultaneously to determine the vibration parameters of a structure. Its software features pre, real-time and post processing tools that allow to prepare and perform impulse and harmonic vibration tests and to calculate modal parameters using specific methods (power complex method, quadrature method). This system also provides graphical representation during and after tests. The equipment is pictured in Fig. 1 with a general view of the data acquisition, control and computing cabinet in (a) and electromechanical shaker connected to the test specimen in (b).

The studied structure is a model wing developed in the frame of a research project for active control of flutter phenomena. The wing has a 1.2m span, a 0.3m chord and is equipped with a piezoelectric-actuated control surface. For the purpose of the vibration test, the control surface has been blocked in the neutral position. The wing has been mounted vertically by clamping it to a very rigid steel base on the ground and excited with an electromechanic shaker mounted in the horizontal direction, as detailed in Fig.1 (b).

(a) (b)

FIG. 1. (a) Prodera equipment, (b) Mechanical link between exciter and wing structure

The wing structure is instrumented with 11 acceleration transducers which are positioned on the wing surface according to Fig. 2 (a), (b) and table 1.

(a) (b)

FIG. 2. (a) Test configuration, (b) Geometry of digitized structure

Table 1. Position of Transducers and Shaker

Point No.	Coord. in x [mm]	Coord. in y [mm]	Comment
1	20	20	Transducer 1
2	20	296	Transducer 2
3	20	572	Transducer 3
4	20	848	Transducer 4
5	20	1124	Transducer 5
6	280	1124	Transducer 6
7	185	848	Transducer 7
8	280	572	Transducer 8

Point No.	Coord. in x [mm]	Coord. in y [mm]	Comment
9	280	296	Transducer 9
10	280	20	Transducer 10
11	145	180	Transducer 11 and Shaker

3. TEST PROCEDURE AND RESULTS

The experimental procedure aimed to identify the modal parameters of the wing structure in three phases, based on phase excitation technique.

Firstly, the structure has been excited from 0 to 50Hz with a 0.1Hz step, and a stabilizing time of 10 seconds. The mode indicator function (MIF), which is based on the fact that at resonance real part of the FRF is zero, was applied on the acquired data. This provided a general view of the number and location of natural frequencies in the interest range 0-50Hz.

Secondly, the structure has been excited around each of these frequencies, in separate tests, in a narrow range (+/- 0.3Hz), with a step of 0.001Hz, and adequate force excitation and stabilizing time. The mode indicator function has been again applied to accurately identify the natural frequencies.

The third phase was mainly automated and consisted in the excitation of the structure around each natural frequency with automatically selected range, step and other parameters. The methods used by Prodera [1] were: automatic complex power (computes natural frequency, modal damping, mass and stiffness associated to the studied mode), automatic logarithmic decrement (computes modal damping), readout of the mode shape (computes mode shape), and phase index (computes the real and imaginary part of the acceleration for the selected frequency).

The test results used for the purpose of this paper are presented in table 2. The mode shapes are graphically represented in Fig.3. The modal mass, damping and stiffness matrices are assembled in tables 3, 4 and 5.

Table 2. Experimental Modal Data

	Mode 1	**Mode 2**	**Mode 3**	**Mode 4**	**Mode 5**
Frequency [Hz]	5.865	14.463	23.137	41.850	49.413
Modal damping	0.00418	0.00108	0.00555	0.00308	0.00847
Modal mass [Kg*m^2]	1.336	4.346	2.181	0.725	3.824
Modal stiffness [Kg*m^2/s^2]	1813.559	35887.128	46094.437	50126.386	368594.781
Transducer	**Mode shapes (not normalized)**				
1	0.000365	5.24E-5	2.89E-5	9.06E-6	3.81E-5
2	0.000987	-8.43E-5	0.001289	-0.00018	0.000708
3	0.002122	-0.00043	0.002048	-0.00034	0.00024
4	0.004253	-0.00053	0.001161	-0.00011	-0.0003
5	0.006077	-0.00071	-0.00077	0.000317	0.0001
6	0.00829	0.001028	-0.00322	-0.00035	0.000624
7	0.005248	0.000396	0.00083	-0.00011	-0.00075
8	0.002785	0.000793	0.00317	0.000383	-0.00062
9	0.000788	0.00016	0.002479	0.000342	0.000778
10	-5.80E-5	6.17E-6	1.61E-5	4.88E-6	2.87E-5
11	0.000464	-1.03E-6	0.000887	2.16E-5	0.000714

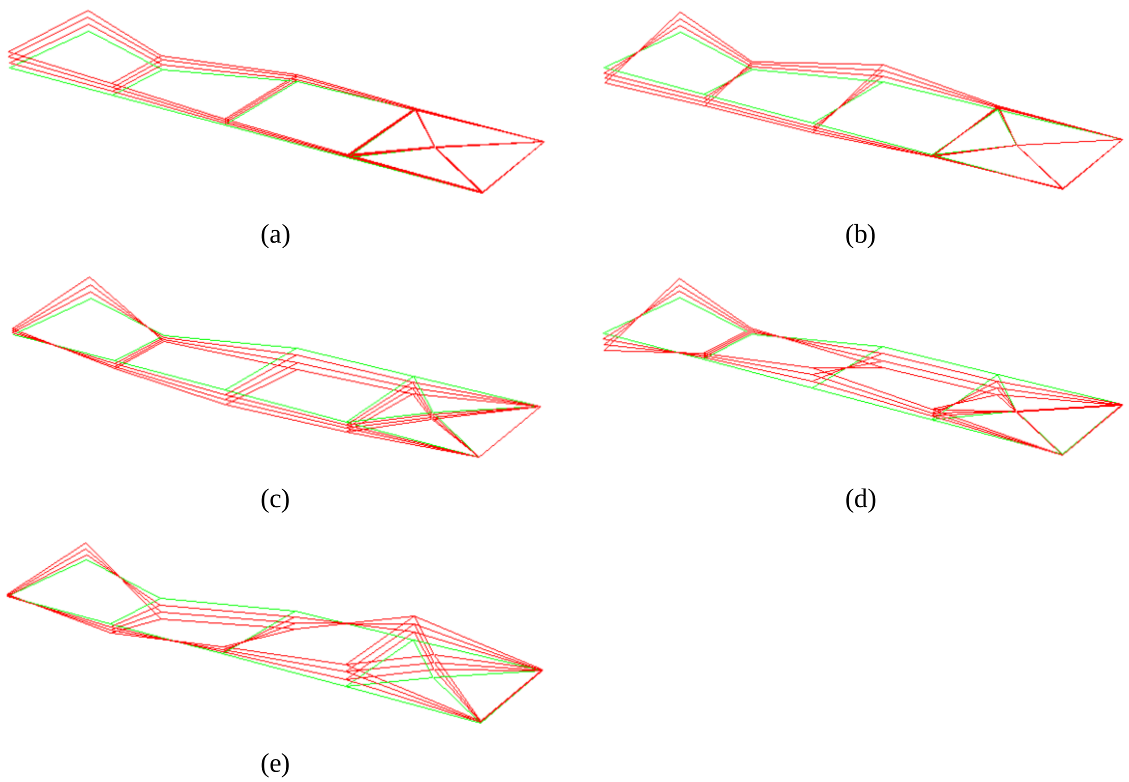

FIG. 3. Mode shapes: (a) Mode 1, (b) Mode 2, (c) Mode 3, (d) Mode 4, (e) Mode 5

Table 3. Modal Mass Matrix

1.336	0	0	0	0
0	4.346	0	0	0
0	0	2.181	0	0
0	0	0	0.725	0
0	0	0	0	3.824

Table 4. Modal Damping Matrix

0.00418	0	0	0	0
0	0.00108	0	0	0
0	0	0.00555	0	0
0	0	0	0.00308	0
0	0	0	0	0.00847

Table 5. Modal Stiffness Matrix

1.81355945E+03	0	0	0	0
0	3.58871289E+04	0	0	0
0	0	4.60944375E+04	0	0
0	0	0	5.01263867E+04	0
0	0	0	0	3.68594781E+05

4. DERIVATION OF [M], [C], [K] MATRICES FROM EXPERIMENTAL MODAL DATA

Once a set of modal data has been measured for the wing structure, it is possible to compute the full mass, stiffness and damping matrices (matrices in spatial coordinates). Formulas derived from the relationship between the time domain differential equations of motion and the transfer function matrix model of the structure are provided in [2]:

$$[M]=\left([\Phi]^t\right)^{-1}[m][\Phi]^{-1} \tag{1}$$

$$[C]=\left([\Phi]^t\right)^{-1}[c][\Phi]^{-1} \tag{2}$$

$$[K]=\left([\Phi]^t\right)^{-1}[k][\Phi]^{-1} \tag{3}$$

where

$[\Phi]=\left[\{u_1\},\{u_2\},...,\{u_m\}\right]$ - mode shape matrix

$[m]$, $[c]$, $[k]$ - modal mass, damping and stiffness matrices (4)

$[M]$, $[C]$, $[K]$ - full mass, damping and stiffness matrices

The direct use of these equations is prohibitively difficult, especially when using experimental modal data, since the mode shape matrix $[\Phi]$ is usually not square (in our case is 11x5).

To overcome this problem, an assumption can be made [2]: if the mode shape vectors are assured to be orthogonal with respect to one another (and are also normalized to unit magnitudes), then the mode shape matrix has the following properties:

$$[\Phi]^t[\Phi]=[I]=[\Phi]^{-1}[\Phi]\Rightarrow[\Phi]^t=[\Phi]^{-1} \tag{5}$$

To take advantage of this properties we used classical Gram-Schmidt algorithm to ortonormalize the mode shape matrix of the wing structure. The result is presented in the next table.

Table 6. Mode Shape Matrix normalized with Gram-Schmidt algorithm

	Mode 1	**Mode 2**	**Mode 3**	**Mode 4**	**Mode 5**
DOF 1	0.028355	2.48E-02	9.25E-03	1.30E-02	2.44E-02
DOF 2	0.076688	-7.15E-02	0.215205	-0.31529	0.381528
DOF 3	0.164976	-0.30742	0.325731	-0.55954	0.075313
DOF 4	0.330597	-0.40826	0.177782	-0.15961	-0.20987
DOF 5	0.472374	-0.55461	-0.15215	0.563064	0.07069
DOF 6	0.644439	0.463578	-0.45139	-0.18477	0.279941
DOF 7	0.40793	0.137127	0.181038	-0.15417	-0.44512
DOF 8	0.216532	0.429949	0.590723	0.312044	-0.2448
DOF 9	0.061222	0.082489	0.429535	0.298114	0.533291
DOF 10	-4.51E-03	4.95E-03	2.88E-03	4.67E-03	1.79E-02
DOF 11	0.036089	-9.97E-03	0.150844	-2.55E-02	0.421877

Checking ortogonality yields very good results:

$$[\Phi]^t[\Phi]=\begin{bmatrix} 1 & 5.93E-17 & -3.73E-17 & 5.35E-17 & 1.56E-17 \\ 5.93E-17 & 1 & -4.16E-17 & 8.19E-17 & 2.34E-17 \\ -3.73E-17 & -4.16E-17 & 1 & -3.56E-17 & -5.55E-17 \\ 5.35E-17 & 8.19E-17 & -3.56E-17 & 1 & 2.43E-17 \\ 1.56E-17 & 2.34E-17 & -5.55E-17 & 2.43E-17 & 1 \end{bmatrix} \cong [I] \tag{6}$$

Because the inverse of the mode shape matrix is now equal to its transpose, the formulas for full mass, stiffness and damping matrices are straightforward [2]:

$$[M]=[\Phi][m][\Phi]^t \tag{7}$$

$$[C]=[\Phi][c][\Phi]^t \tag{8}$$

$$[K]=[\Phi][k][\Phi]^t \tag{9}$$

The resulted matrices are presented in tables 7, 8 and 9:

Table 7. Mass Matrix in spatial coordinates

0.006341	3.22E-02	-1.86E-02	-4.90E-02	-3.31E-02	0.0897	-0.00914	0.046552	0.072513	0.002137	0.04251
3.22E-02	7.60E-01	5.03E-01	-2.55E-02	1.24E-01	0.160713	-0.53003	-0.26269	0.892146	0.024403	0.698933
-1.86E-02	5.03E-01	0.927163	7.49E-01	5.29E-01	-0.64242	-0.03033	-0.30413	0.241087	-0.0023	0.260285
-4.90E-02	-2.55E-02	7.49E-01	1.126217	1.01E+00	-0.91619	0.382137	-0.27781	-0.41526	-0.02456	-0.24351
-3.31E-02	1.24E-01	5.29E-01	1.01E+00	1.934379	-0.56065	-0.31642	-1.0345	-0.03688	-0.00899	0.100368
0.0897	0.160713	-0.64242	-0.91619	-0.56065	2.257634	-0.00659	0.167238	0.326983	0.021785	0.317522
-0.00914	-0.53003	-0.03033	0.382137	-0.31642	-0.00659	1.150406	0.989282	-0.68893	-0.02936	-0.64195
0.046552	-0.26269	-0.30413	-0.27781	-1.0345	0.167238	0.989282	1.926845	0.293468	-0.00404	-0.21454
0.072513	0.892146	0.241087	-0.41526	-0.03688	0.326983	-0.68893	0.293468	1.58895	0.041624	0.995513
2.14E-03	2.44E-02	-2.30E-03	-2.46E-02	-8.99E-03	0.021785	-0.02936	-0.00404	0.041624	0.001393	0.02931
0.04251	6.99E-01	0.260285	-2.44E-01	0.100368	0.317522	-0.64195	-0.21454	0.995513	0.02931	0.732868

Table 8. Damping Matrix in spatial coordinates

1.01E-05	8.46E-05	2.12E-05	-1.24E-05	7.05E-05	0.000116	-3.70E-05	2.94E-05	0.000154	3.64E-06	9.80E-05
8.46E-05	1.83E-03	1.25E-03	-1.73E-04	-3.06E-04	0.000716	-0.00095	-0.00035	0.00196	5.49E-05	0.001581
2.12E-05	1.25E-03	0.001817	8.26E-04	-6.90E-04	-2.85E-05	0.000545	0.000381	0.000618	3.83E-06	0.000614
-1.24E-05	-1.73E-04	8.26E-04	0.001264	3.45E-04	-0.00017	0.001549	0.000974	-0.00062	-3.97E-05	-0.00053
7.05E-05	-3.06E-04	-6.90E-04	3.45E-04	0.002412	0.001223	3.66E-05	6.58E-05	0.000545	4.52E-06	0.000158
0.000116	0.000716	-2.85E-05	-0.00017	0.001223	0.003868	-0.00025	-0.00144	0.000225	2.29E-05	0.000729
-3.70E-05	-0.00095	0.000545	0.001549	3.66E-05	-0.00025	0.002649	0.001801	-0.0016	-7.38E-05	-0.00137
2.94E-05	-0.00035	0.000381	0.000974	6.58E-05	-0.00144	0.001801	0.00314	0.000683	-2.50E-05	-0.00038
0.000154	0.00196	0.000618	-0.00062	0.000545	0.000225	-0.0016	0.000683	0.00373	9.13E-05	0.00225
3.64E-06	5.49E-05	3.83E-06	-3.97E-05	4.52E-06	2.29E-05	-7.38E-05	-2.50E-05	9.13E-05	2.94E-06	6.53E-05
9.80E-05	1.58E-03	0.000614	-5.34E-04	0.000158	0.000729	-0.00137	-0.00038	0.00225	6.53E-05	0.001641

Table 9. Stiffness Matrix in spatial coordinates

2.56E+02	3.26E+03	1.88E+02	-2.26E+03	4.69E+02	2653.935	-3.89E+03	-1.36E+03	5256.299	1.70E+02	3.84E+03
3.26E+03	6.10E+04	2.35E+04	-2.41E+04	1.02E+03	36710.03	-58659.5	-34571.1	74342.57	2.46E+03	61258.56
1.88E+02	2.35E+04	26116.03	5.92E+03	-9.85E+03	1.25E+03	-6704.87	-11356.9	12000.22	3.53E+02	14812.64
-2.26E+03	-2.41E+04	5.92E+03	25148.67	-2.81E+03	-30281.9	35385.82	15111.85	-41291.1	-1.47E+03	-31027.4
4.69E+02	1.02E+03	-9.85E+03	-2.81E+03	30244.51	-3429.85	-1.96E+04	-1.01E+04	17707.6	4.76E+02	9443.556
2653.935	36710.03	1.25E+03	-30281.9	-3429.85	48454.39	-45510.2	-33034.6	44773.21	1.82E+03	40505.5
-3.89E+03	-58659.5	-6704.87	35385.82	-1.96E+04	-45510.2	76708.61	44957.64	-85764.4	-2.93E+03	-67783.1
-1.36E+03	-34571.1	-11356.9	15111.85	-1.01E+04	-33034.6	44957.64	49773.19	-30464	-1.39E+03	-34498
5256.299	74342.57	12000.22	-41291.1	17707.6	44773.21	-85764.4	-30464	118038.4	3.66E+03	85507.52
1.70E+02	2.46E+03	3.53E+02	-1.47E+03	4.76E+02	1.82E+03	-2.93E+03	-1.39E+03	3.66E+03	1.21E+02	2.80E+03
3.84E+03	6.13E+04	14812.64	-3.10E+04	9443.556	40505.5	-67783.1	-34498	85507.52	2.80E+03	66690.13

5. VALIDATION OF [M], [C], [K] MATRICES

In absence of damping and external load, the equation of motion in matrix form for a given structure is [3]:

$$[M]\{\ddot{q}\}+[K]\{q\}=0 \tag{10}$$

which, for solving, we assume a harmonic solution of the form:

$$\{q\}=\{u\}\sin\varpi t \tag{11}$$

Through differentiation and substitution in equation 10, the following is obtained:

$$-\varpi^2[M]\{u\}\sin\varpi t+[K]\{u\}\sin\varpi t=0 \tag{12}$$

which after simplifying becomes:

$$\left([K]-\varpi^2[M]\right)\{u\}=0 \tag{13}$$

This reduces to an eigenvalue problem of the basic form:

$$[A-\lambda I]x=0 \tag{14}$$

By solving the eigenvalues problem we find the eigenvalues and eigenvectors associated to the structure. The ith eigenvalue λ_i is related to the ith natural frequency f_i as follows:

$$f_i=\frac{\varpi_i}{2\pi},\quad \varpi_i=\sqrt{\lambda_i} \tag{15}$$

The natural frequencies f_i have been computed using mass [M] and stiffness [K] matrices, previously derived for the wing structure, and the results are presented in the next table along with the experimentally determined frequencies. Very good correlation is found.

Table 10. Computed and experimental frequencies

No.	Computed frequencies	Experimental frequencies	No.
1	5.86384930583323	5.865	1
2	14.4625418258649	14.463	2
3	16.1181473527133	-	
4	21.5627768416534	-	
5	23.1375067594382	23.137	3
6	35.1286393566203	-	
7	37.3453361756750	-	
8	41.8489271158046	41.850	4
9	45.8132201895832	-	
10	49.4123809131585	49.413	5
11	49.7003108389488	-	

CONCLUSIONS

The experimentally determined frequencies are all found through the numerical procedure with very good accuracy. This confirms that [M] and [K] matrices accurately describe the wing structure (up to the fifth vibration mode). This also confirms that [C] matrix, which was neglected in equation 10, is indeed negligible, in line with the small values presented in table 8.

As can be seen in table 10, the computed frequencies no. 3,4,6,7,9 and 11 do not have correspondence in the set of experimental frequencies. The experimental modal analysis was performed in a very accurate manner and no other natural frequency has been found between the experimental frequencies listed above. The values of these additional (computed) frequencies vary with the quality of the mode shape matrix orthonormalization (for example by using classical Gram-Schmidt or modified Gram-Schmidt methods) and are a direct consequence of constructing 11x11 mass, stiffness and damping matrices from experimental modal data gathered from only 5 modes of vibration.

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The tests were performed at STRAERO facilities.

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