OSCILLATORY SOLUTIONS FOR THE SYSTEMS OF DIFFERENCE EQUATIONS WITH VARIABLE COEFFICIENTS

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Abstract*: In this paper we give some sufficient conditions for the oscillation of all solutions of the following system of neutral difference equations with variable coefficients*

$$\Delta(u_i(n)+cu_i(n\text{-}a\lambda))+\sum_{j=1}^{r}\alpha_{ij}(n)\,u_j(n\text{-}k)=0\ ;\ i=1,2,...,r$$

where $\alpha_{ij}(n)$ *are real sequences with* $i,j=1,2,...,r$; $\lambda,k\in Z^+$, $a=\pm1$ *and* $c\in[-1,\infty)$.

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1. INTRODUCTION AND PRELIMINARIES

In this paper, we shall look at the oscillatory properties of all the solutions of the system of neutral difference equations with variable coefficients

$$\Delta(u_i(n)+cu_i(n\text{-}a\lambda))+\sum_{j=1}^{r}\alpha_{ij}(n)\,u_j(n\text{-}k)=0$$

$$i=1,2,...,r \tag{1}$$

where $\alpha_{ij}(n)$ are real sequences with $i,j=1,2,...,r$ and $a=\pm1$, λ,k are positive integers, $c\in[-1,\infty)$ and Δ is the first order forward difference operator, i.e,

$$\Delta u(n)=u(n+1)-u(n).$$

Definition 1. We say that a solution $u(n)=[u_1(n),u_2(n),...,u_r(n)]^t$ of equation (1) is oscillating if for some $i\in\{1,2,...,r\}$ and for every integer $n_0>0$, there exists $n>n_0$ such that $u_i(n)u_i(n+1)<0$.

Definition 2. We say that a solution $u(n)=[u_1(n),...,u_r(n)]^t$ is nonoscillatory if it is not eventually the trivial solution and if each component $u_i(n)$ has eventually constant signum.

Although the problem oscillation solutions for the difference equations has attracted many researchers, in recent years there has been much research activity concerning the oscillation of solutions of delay difference equations. For these oscillatory results, we refer to the [1,2,3,4,5,6] and the references therein.

In [2] Agarwal and Grace established oscillation criteria for the higher order systems of difference equations with constant coefficients. Further, in [3] Chuanxi, Kuruklis and Ladas studied oscillatory behaviour of systems of difference equations with variable coefficients.

In this paper, we obtain sufficient conditions for the oscillations of all the solutions of (1). To establish the main results we need a result on the oscillation of solutions of an Eq. with regressive differences [5].

Lemma 1. Let k be a positive integer and let $\{\alpha_n\}$ be a sequence of non-negative real numbers such that

$$\sum_{j=0}^{k}\alpha_{n+j}>0 \tag{2}$$

for all large n.

Assume that $\{v_n\}$ is a solution of the following difference inequalities

$$v_{n+1} - v_n + \alpha_n v_{n-k} \le 0,\ n = 0,1,2,... \quad (3)$$

such that

$$v_n > 0 \text{ for } n \ge -k,$$

Then the difference equation

$$u_{n+1} - u_n + \alpha_n u_{n-k} = 0,\ n = 0,1,2,... \quad (4)$$

has a solution $\{u_n\}$ such that $0 < u_n \le v_n$ for $n \ge -k$ and

$$\lim_{n\to\infty} u_n = 0 \quad (5)$$

Lemma 2. Suppose that $\{\alpha_n\}$ is a positive sequence of real numbers and let k be a positive integer. For every solution of equation (4) to be oscillatory is sufficient to have the relationship

$$\liminf_{n\to\infty}\left[\frac{1}{k}\sum_{i=n-k}^{n-1}\alpha_i\right] > \frac{k^k}{(k+1)^{k+1}} \quad (6)$$

Proof. Assume, for the sake of contradiction, that equation (4) has a nonoscillatory solution $\{u_n\}$. As the opposite of a solution of Eq. (4) is also a solution, we may (and do) assume that $\{u_n\}$ is eventually positive. Then eventually

$$u_{n+1} - u_n = -\alpha_n u_{n-k} \le 0$$

and so $\{u_n\}$ is an eventually decreasing sequence of positive numbers. It follows from equation (4) that eventually

$$u_{n+1} - u_n + \alpha_n u_{n-k} \le 0$$

or

$$\alpha_n \le 1 - \frac{u_{n+1}}{u_n}$$

and so eventually,

$$\frac{1}{k}\sum_{i=n-k}^{n-1}\alpha_i \le \frac{1}{k}\sum_{i=n-k}^{n-1}\left(1 - \frac{u_{n+1}}{u_n}\right) \quad (7)$$

Set

$$\alpha = \frac{k^k}{(k+1)^{k+1}} \quad (8)$$

then, from (6), it is clear that we can choose a constant β such that, for n sufficiently large,

$$\alpha < \beta \le \frac{1}{k}\sum_{i=n-k}^{n-1}\alpha_i \quad (9)$$

Thus, in view of (7),

$$\beta \le \frac{1}{k}\sum_{i=n-k}^{n-1}\left(1 - \frac{u_{n+1}}{u_n}\right) \quad (10)$$

for all large n.

By using (10) and the well-known inequality between the arithmetic and geometric means we find that for n sufficiently large,

$$\beta \le \frac{1}{k}\sum_{i=n-k}^{n-1}\left(1 - \frac{u_{i+1}}{u_i}\right) = 1 - \frac{1}{k}\sum_{i=n-k}^{n-1}\frac{u_{i+1}}{u_i} \le$$

$$\le 1 - \left(\prod_{i=n-k}^{n-1}\frac{u_{i+1}}{u_i}\right)^{1/k} = 1 - \left(\frac{u_n}{u_{n-k}}\right)^{1/k}$$

that is,

$$\left(\frac{u_n}{u_{n-k}}\right)^{1/k} \le 1 - \beta$$

(11)

for all large n.

In particular, this implies that $0 < \beta < 1$. Now observe that

$$\max_{0\le\lambda\le1}\left[(1-\lambda)\lambda^{1/k}\right] = \frac{k}{(k+1)^{1/k}} = \alpha^{1/k}$$

where α is the positive constant defined by (8).

Therefore

$$1 - \lambda \le \alpha^{1/k}\lambda^{-1/k} \text{ for } 0 < \lambda \le 1$$

and (11) yields

$$\frac{\beta}{\alpha}u_n \le u_{n-k} \quad (12)$$

for all large n.

By using (12) in Eq. (4) and then by repeating the above arguments we find that

$$\left(\frac{\beta}{\alpha}\right)^2 u_n \le u_{n-k} \text{ for all large n}$$

and, by induction, for every $m = 1,2,...$ there exists an integer N_m such that for $n \ge N_m$

$$\left(\frac{\beta}{\alpha}\right)^m u_n \le u_{n-k} \qquad (13)$$

Next observe that because of (9), for n sufficiently large,

$$\sum_{i=n-k}^{n}\alpha_i \ge \sum_{i=n-k}^{n-1}\alpha_i \ge k\beta$$

Hence, for n sufficiently large,

$$\sum_{i=n-k}^{n}\alpha_i \ge \beta \qquad (14)$$

Where $B = k\beta$. Choose m such that

$$\left(\frac{\beta}{\alpha}\right)^m > \left(\frac{2}{B}\right)^2 \qquad (15)$$

This is possible because from (9), $\beta > \alpha$. Then for n sufficiently large, say for $n \ge n_0$, (13) is satisfied for the specific m which was chosen in (15), also (9) and (14) hold, and $\{u_n\}$ is decreasing for $n > n_0$. Now in view of (14) and for $n \ge n_0 + k$, there exists an integer $\tilde{n}$ with $n - k \le \tilde{n} \le n$ such that

$$\sum_{i=n-k}^{\tilde{n}}\alpha_i \ge \frac{\beta}{2} \text{ and } \sum_{i=\tilde{n}}^{n}\alpha_i \le \frac{\beta}{2}$$

From equation (4) and the decreasing nature of $\{u_n\}$, we have

$$u_{\tilde{n}-1} - u_{n-k} = \sum_{i=n-k}^{\tilde{n}}(u_{i+1} - u_i) = -\sum_{i=n-k}^{\tilde{n}}\alpha_i u_{i-k}$$

$$\le -\left(\sum_{i=n-k}^{\tilde{n}}\alpha_i\right)u_{\tilde{n}-k} \le -\frac{\beta}{2}u_{\tilde{n}-k}$$

hence,

$$\frac{\beta}{2}u_{\tilde{n}-k} \le u_{n-k} \qquad (16)$$

Similarly,

$$u_{n+1} - u_n = \sum_{i=\tilde{n}}^{n}(u_{i+1} - u_i) = -\sum_{i=\tilde{n}}^{n}\alpha_i u_{i-k}$$

$$\le -\left(\sum_{i=\tilde{n}}^{n}\alpha_i\right)u_{n-k} \le -\frac{\beta}{2}u_{n-k}$$

and so

$$\frac{\beta}{2}u_{n-k} \le u_{\tilde{n}} \qquad (17)$$

From (16) and (17) we find

$$\left(\frac{\beta}{2}\right)^2 u_{\tilde{n}-k} \le u_{\tilde{n}}$$

which in view of (13) yields

$$\left(\frac{\beta}{\alpha}\right)^m \le \frac{u_{\tilde{n}-k}}{u_{\tilde{n}}} \le \left(\frac{2}{M}\right)^2$$

This contradicts (14) and so the proof of the theorem is complete.

2. MAIN RESULTS

In this section, we shall establish a few sufficient conditions for the oscillations of all the solutions of equations (1).

i. First, we analyze the behavior of the solutions oscillating system with variable coefficients (1) for a while $c = 0$.

$$u_i(n+1) - u_i(n) + \sum_{j=1}^{r}\alpha_{ij}(n)\,u_j(n\text{-}k) = 0 \qquad (18)$$

where $\{\alpha_{ij}(n)\}$ are real sequences with $i, j = 1,2,...,r$ and $k \in Z^+$. In this sense we can state the following Theorem.

Theorem 1. Suppose that $c = 0$. Let $\{\alpha_{ij}(n)\}$ be real sequences with $i, j = 1,2,...,r$ and let K be a positive integer. If every solution of the equation

$$v(n+1) - v(n) + \alpha(n)v(n-k) = 0 \qquad (19)$$

oscillates, where

$$\alpha(n) = \min_{1\le i\le r}\left\{\alpha_{ii}(n) - \sum_{j=1, j\ne i}^{r}\left|\alpha_{ji}(n)\right|\right\} > 0 \qquad (20)$$

then every solution of (18) oscillates.

Proof. Assume that equation (18) has a nonoscillatory and eventually positive solution $u(n) = [u_1(n), u_2(n),..., u_r(n)]^t$. Then, there exists an integer $n_0 > 0$ such that $u_i(n) > 0$ for $n \ge n_0$, $i = 1,2,...,r$. If we let

$$w(n) = \sum_{j=1}^{r}u_j(n)$$

then

$$w(n+1) - w(n) = -\sum_{i=1}^{r}\alpha_{ii}(n)u_i(n-k) -$$

$$-\sum_{i=1}^{r}\sum_{j=1,j\neq i}^{r}\alpha_{ij}(n)u_j(n-k)\leq -\sum_{i=1}^{r}\alpha_{ii}(n)u_i(n-k)+$$
$$+\sum_{i=1}^{r}\sum_{j=1,j\neq i}^{r}\left|\alpha_{ij}(n)\right|u_j(n-k)$$

Therefore, from the above inequality we find the following

$$w(n+1)-w(n)+$$
$$+\sum_{i=1}^{r}\left[\alpha_{ii}(n)-\sum_{j=1,j\neq i}^{r}\left|\alpha_{ji}(n)\right|\right]u_i(n-k)\leq 0$$

or

$$w(n+1)-w(n)+\alpha(n)w(n-k)\leq 0) \quad (21)$$
$$n\geq n_1\geq n_0$$

By the eventual positivity of $u_1(n)$, $u_2(n)$, $\ldots, u_r(n)$ we conclude that w(n) is eventually positive. Then by Lemma 1, we see that

$$v(n+1)-v(n)+\alpha(n)v(n-k)=0$$

has a positive solution $\{v(n)\}$ for $n\geq n_1$, which contradicts our hypothesis and completes the proof. Thus, we have the following corollary that is immediate.

Corollary 1. Let $\alpha(n)$ be as in (20) and k be a positive integer. If

$$\lim_{n\to\infty}\inf\left[\frac{1}{k}\sum_{i=n-k}^{n-1}\alpha(i)\right]>\frac{k^k}{(k+1)^{k+1}}$$

holds, then all the solutions of the equation (1) oscillate.

Proof. It follows immediately if one takes into account the Lemma 2 and Theorem 1.

Suppose now that $c\in[-1,0)\cup(0,\infty)$, we can state the following Theorem.

Theorem 2. Let $\{\alpha_{ij}(n)\}$ be real sequences with $i,j=1,2,\ldots,r$; λ,k $(k>\lambda)$ are positive integers and $\alpha(n)$ is defined in (20).

a. If $0<c<1$, $a=-1$ and every solution of the equation

$$v(n+1)-v(n)+(1-c)\alpha(n)v(n-k)=0$$
(22)

is oscillatory, then every solution of equation (1) is oscillating.

b. If $c>1, a=1$ and every solution of the equation

$$v(n+1)-v(n)+$$
$$+\frac{1-c}{c^2}\alpha(n)v(n-(k-\lambda))=0 \quad (23)$$

is oscillatory, then every solution of equation (1) is oscillating.

c. If $c=1$, $a=-1$ and every solution of the equation

$$v(n+1)-v(n)+\frac{1}{2}\alpha(n)v(n-k)=0 \quad (24)$$

is oscillatory, then every solution of equation (1) is oscillating.

d. If $-1\leq c<0$, $a=-1$ and every solution of the equation

$$v(n+1)-v(n)+\alpha(n)v(n-k)=0 \quad (25)$$

is oscillatory, then every solution of equation (1) is oscillating.

Proof. Suppose that $u(n)=[u_1(n),\ldots,u_r(n)]^t$ is a nonoscillatory and eventually positive solution of (1), $a=\pm1$. Then, there exists an integer $n_0\geq 0$ such that $u_i(n)>0$ for $n\geq n_0$, $i=1,2,\ldots,r$.

We let

$$v(n)=\sum_{i=1}^{r}u_i(n)+c\sum_{i=1}^{r}u_i(n-a\lambda) \quad (26)$$

then,we have

$$v(n+1)-v(n)=\sum_{i=1}^{r}\Delta(u_i(n)+cu_i(n-a\lambda))=$$
$$=-\sum_{i=1}^{r}\sum_{j=1}^{r}\alpha_{ij}(n)u_j(n-k)$$

or

$$v(n+1)-v(n)+\sum_{i=1}^{r}\sum_{j=1}^{r}\alpha_{ij}(n)u_j(n-k)$$

So, as in Theorem 1, we have for $n\geq n_1\geq n_0$

$$v(n+1)-v(n)+\alpha(n)w(n-k)\leq 0 \quad (27)$$

where

$$w(n)=\sum_{i=1}^{r}u_i(n)$$

It is clear that $\{v(n)\}$ and $\{w(n)\}$ are positive sequences. We see from (26) that if $a=-1$ and $0<c<1$, then eventually

$$v(n)=w(n)+cw(n+\lambda),$$

and we get eventually,

$$w(n) = v(n) - cw(n+\lambda) \geq$$
$$\geq v(n) - cv(n+\lambda) \geq (1-c)v(n)$$

therefore, we get eventually,

$$w(n-k) \geq (1-c)v(n-k) \qquad (28)$$

If $c > 1$ and $a = 1$, then,

$$w(n) = \frac{1}{c}(v(n+\lambda) - w(n+\lambda)) =$$
$$= \frac{1}{c}v(n+\lambda) - \frac{1}{c^2}(v(n+2\lambda) - w(n+2\lambda)) \geq$$
$$\geq \frac{1}{c}v(n+\lambda) - \frac{1}{c^2}v(n+\lambda) =$$
$$= \frac{c-1}{c^2}v(n+\lambda)$$

therefore, by using above inequlity, we get eventually,

$$w(n-k) \geq \frac{c-1}{c^2}v(n-(k-\lambda)) \qquad (29)$$

Now, we take the $c = 1$ and $a = -1$. Then, by (26) eventually,

$$v(n) = w(n) + w(n+\lambda)$$

so eventually,

$$w(n) = v(n) - w(n+\lambda) \geq v(n) - w(n)$$

and we have eventually,

$$w(n) \geq \frac{1}{2}v(n) \qquad (30)$$

Now, we take the $-1 \leq c < 0$ and $a = -1$. Then, by (26) eventually,

$$v(n) = w(n) + cw(n+\lambda)$$

and we have eventually,

$$w(n) = v(n) - cw(n+\lambda)$$

and so eventually,

$$w(n) \geq v(n)$$

and we have eventually,

$$w(n-k) \geq v(n-k) \qquad (31)$$

Next, from the above we have the following:

a. If $0 < c < 1$, and $a = -1$, by (31) and (28), we obtain eventually,

$$v(n+1) - v(n) + (1-c)\alpha(n)v(n-k) \leq 0$$

b. If $c > 1$, and $a = 1$, then, by (27) and (29), we obtain eventually,

$$v(n+1) - v(n) + \frac{1-c}{c^2}\alpha(n)v(n-(k-\lambda)) \leq 0$$

c. If $c = 1$, and $a = -1$, then, by (27) and (30), we obtain eventually,

$$v(n+1) - v(n) + \frac{1}{2}\alpha(n)v(n-k) \leq 0$$

d. If $-1 \leq c < 0$ and $a = -1$, then, by (27) and (31), we obtain eventually,

$$v(n+1) - v(n) + \alpha(n)v(n-k) \leq 0$$

Using the same reasoning as in Theorem 1 we arrive at a contradiction in each of the above situations and the theorem is completely proven.

The following corollaries are immediate, if one takes into account Lemma 2 and Theorem 2.

Corollary 2. Let $\{\alpha_{ij}(n)\}$ be real sequences with $i, j = 1,2,...,r$; λ, k $(k > \lambda)$ be positive integers and $\alpha(n)$ is defined in (20).

a. Suppose that $0 < c < 1$, $a = -1$, if

$$\liminf_{n\to\infty}\left[\frac{1}{\lambda}\sum_{i=n-\lambda}^{n-1}\alpha(i)\right] > \frac{1}{1-c}\frac{\lambda^\lambda}{(\lambda+1)^{\lambda+1}}$$

then every solution of (1) is oscillating.

b. Suppose that $c > 1, a = 1$, if

$$\liminf_{n\to\infty}\left[\frac{1}{\lambda}\sum_{i=n-\lambda}^{n-1}\alpha(i)\right] > \frac{c^2}{c-1}\frac{(k-\lambda)^\lambda}{(k-\lambda+1)^{k-\lambda+1}}$$

then every solution of (1) is oscillating.

c. Suppose that $c = 1$, $a = -1$, if

$$\liminf_{n\to\infty}\left[\frac{1}{\lambda}\sum_{i=n-\lambda}^{n-1}\alpha(i)\right] > 2\frac{\lambda^\lambda}{(\lambda+1)^{\lambda+1}}$$

then every solution of (1) is oscillating.

d. Suppose that $-1 \leq c < 0$, $a = -1$, if

$$\liminf_{n\to\infty}\left[\frac{1}{\lambda}\sum_{i=n-\lambda}^{n-1}\alpha(i)\right] > \frac{\lambda^\lambda}{(\lambda+1)^{\lambda+1}}$$

then every solution of (1) is oscillating.

4. CONCLUSIONS

The results of this work may constitute the starting point for other generalizations. In reference [6], under some appropriate conditions over the real sequences $\{\alpha_n\}$, and $\{\beta_n\}$, the behavior of all the solutions of the oscillating difference equation with variable coefficients is studied.

$$u_{n+1} - u_n + \sum_{i=1}^{r} \alpha_{in} u_{n-k} + \beta_n u_{n-m} = 0$$

Where
$m \in \{\ldots,-2,-1,0\}$, $k_i \in N$ and $k_i \in \{\ldots -3,-2\}$, $i = 1,2,\ldots,r$, respectively.

It is therefore, interesting to analyze the oscillating behavior of the system of the difference equations with variable coefficients corresponding to

$$u_i(n+1) - u_i(n) + \sum_{j=1}^{r} \alpha_{ij}(n)\, u_j(n-k_j) +$$
$$+ \beta_i(n) u_i(n-m) = 0,\ \ i = 1,2,\ldots,r$$

under certain conditions imposed on coefficients $\alpha_{ij}(n)$ and $\beta_i(n)$, eg

$$\liminf_{n\to\infty} \alpha_{ij}(n) = \alpha_{ij}$$

and

$$\liminf_{n\to\infty} \beta_i(n) = \beta_i > 0.$$

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